ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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SUMMARY

In a recent work, we presented a new technique for analyzing the asymptotic nature of solutions of linear differential equations and linear differential-difference equations. We mentioned there that the same method is applicable whenever a certain integral representation, essentially a convolution transform, exists for the solution. The purpose of this note is to initiate the application of the method to the study of partial differential equations possessing this property.

Here we consider the parabolic equation

$$u_t = u_{xx} + \phi(x)a(t)u,$$

with the boundary conditions $u(0,t) = u(1,t) = 0$, $t > 0$. In subsequent papers, we shall apply the method to other types of functional equations. For the sake of simplicity, assume that $a(t) = (t + 1)^{-1}$ or $(t + 1)^{-2}$, and that $\phi(x)$ is a continuous function over $0 \leq x \leq 1$.

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1. INTRODUCTION

In a recent work[1], we presented a new technique for analyzing the asymptotic nature of solutions of linear differential equations and linear differential-difference equations. We mentioned there that the same method is applicable whenever a certain integral representation, essentially a convolution transform, exists for the solution. The purpose of this note is to initiate the application of the method to the study of partial differential equations possessing this property.

Here we consider the parabolic equation

$$u_t = u_{xx} + \phi(x)a(t)u, \tag{1}$$

with the boundary conditions $u(0,t) = u(1,t) = 0$, $t > 0$. In subsequent papers, we shall apply the method to other types of functional equations. For the sake of simplicity, assume that $a(t) = (t+1)^{-1}$ or $(t+1)^{-2}$, and that $\phi(x)$ is a continuous function over $0 \leq x \leq 1$. Then we can establish

Theorem. *For each positive integer* n, *let*

$$c_n = \int_0^1 \phi(x_1)\sin^2 n\pi x_1 dx_1. \tag{2}$$

If $a(t) = (t+1)^{-2}$, *for each* n *there is a solution of* (1), *satisfying the stated boundary conditions, with the asymptotic form*

$$u(x,t) = e^{-n^2\pi^2 t}\sin n\pi x + O(e^{-n^2\pi^2 t}t^{-1}) \tag{3}$$

as $t \to \infty$.

If $a(t) = (t+1)^{-1}$, *there is a solution of the form*

$$u(x,t) = e^{-n^2\pi^2 t}\sin n\pi x + o(e^{-n^2\pi^2 t}) \tag{4}$$

if $c_n \leq 0$, *and of the form*

$$u(x,t) = e^{-n^2\pi^2 t} t^{2c_n} \sin n\pi x + o(e^{-n^2\pi^2 t} t^{2c_n}) \qquad (5)$$

if $c_n > 0$. The implicit constants in the o-relations are uniform in x for $0 \leq x \leq 1$.

It will be clear from what follows how more general forms of $a(t)$ are handled.

2. CONVOLUTION-TYPE INTEGRAL EQUATION

It is easy to see that the solution of

$$u_t = u_{xx} + f(x,t),$$
$$u(x,0) = 0, \quad 0 \leq x \leq 1, \qquad (1)$$
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

may be written in the form

$$u = \int_0^t \int_0^1 k(x,x_1,t - t_1)f(x_1,t_1)dx_1dt_1, \qquad (2)$$

where

$$k(x,x_1,s) = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 s}\sin n\pi x \sin n\pi x_1 \qquad (3)$$
$$= \theta_3(\pi(x - x_1)/2, e^{-\pi^2 t}) - \theta_3(\pi(x + x_1)/2, e^{-\pi^2 t})$$

in the notation of Whittaker and Watson[2].

Observe the convolutary nature of the integral.

3. NON-ABSOLUTELY INTEGRABLE $a(t)$

Since the case where $a(t)$ is absolutely integrable over $[0,\infty]$ is rather simple to treat, let us concentrate on the more interesting and difficult case where this is not so. Let us take $a(t) = (t + 1)^{-1}$ as an example, $n = 1$, and c_1, as defined by (1.2), to be nonnegative.

Applying the result of (2.2) to (1.1), with $f(x,t) = \phi(x)a(t)u$, we obtain a linear integral equation which we write in the disjointed form

$$u(x,t) = e^{-\pi^2 t} \sin \pi x + p(x,t) \tag{1}$$
$$+ 2 \sin \pi x \int_0^t \int_0^1 e^{-\pi^2(t-t_1)} \sin \pi x_1 \phi(x_1)(t_1+1)^{-1} u(x_1,t_1)dx_1 dt_1$$

where

$$p(x,t) = \int_0^t \int_0^1 k_1(x,x_1,t-t_1)\phi(x_1)(t_1+1)^{-1} u(x_1,t_1)dx_1 dt_1, \tag{2}$$
$$k_1(x,x_1,t) = 2 \sum_{n=2}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x \sin n\pi x_1.$$

The expression in (1) suggests that we introduce a new variable

$$v(t) = \int_0^1 \phi(x_1) \sin \pi x_1 u(x_1,t_1)dx_1, \tag{3}$$

and write

$$u(x,t) = e^{-\pi^2 t} \sin \pi x + p(x,t)$$
$$+ 2 \sin \pi x \int_0^t e^{-\pi^2(t-t_1)}(t_1 + 1)^{-1} v(t_1)dt_1. \tag{4}$$

Multiplying by $\phi(x) \sin \pi x$ and integrating over x, we derive the relation

$$v(t) = c_1 e^{-\pi^2 t} + 2c_1 \int_0^t e^{-\pi^2(t-t_1)}(t_1 + 1)^{-1} v(t_1)dt_1 + q(t), \tag{5}$$

where

$$q(t) = \int_0^1 p(x,t)\phi(x) \sin \pi x dx. \tag{6}$$

Following the procedure used in [1], we replace (5) by a more tractable integral equation. Differentiating (5), we obtain

$$v' - q' = (-\pi^2 + 2c_1/(t + 1))(v - q) + 2c_1 q/(t + 1), \tag{7}$$

which yields the relation

$$v(t) = c_1 e^{s(t)} + 2c_1 e^{s(t)} \int_0^t e^{-s(t_1)}(t_1 + 1)^{-1} q(t_1)dt_1 + q(t), \tag{8}$$

where $s(t) = \int_0^t (-\pi^2 + 2c_1/(t_1 + 1))dt_1.$

This equation, together with the equations relating u, v, p and q, can be used to derive the stated result. The same method can be used to handle more general functions $a(t)$.

1. R. Bellman and K. L. Cooke, "Asymptotic behavior of solutions of differential-difference equations," Memoirs Amer. Math. Soc., No. 35, 1959.

2. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, New York, 1946, Chapter 21.